

# Cohomology of $\mathfrak{aff}(1)$ and $\mathfrak{aff}(1|1)$ acting on the space of $n$ -ary differential operators on the superspace $\mathbb{R}^{1|1}$

Mabrouk Ben Ammar      Maha Boujelben      Amina Jabeur      Rabeb Sidaoui \*

May 24, 2016

## Abstract

We consider the  $\mu$ -densities spaces  $\mathcal{F}_\mu$  with  $\mu \in \mathbb{R}$ , we compute the space  $H_{\text{diff}}^1(\mathfrak{aff}(1), D_{\lambda,\mu})$  where  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$  and  $D_{\lambda,\mu}$  is the space of  $n$ -ary differential operators from  $\mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n}$  to  $\mathcal{F}_\mu$ . We also compute the super analog space  $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda,\mu})$ .

**Mathematics Subject Classification** 17B56, 17B10, 17B66

**Key words** : Cohomology, Superalgebra, Weighted densities.

## 1 Introduction

The space of weighted densities of weight  $\mu$  on  $\mathbb{R}$  (or  $\mu$ -densities for short), denoted by:

$$\mathcal{F}_\mu = \{f dx^\mu, f \in C^\infty(\mathbb{R})\}, \quad \mu \in \mathbb{R},$$

is the space of sections of the line bundle  $(T^*\mathbb{R})^{\otimes \mu}$ . The Lie algebra  $\text{Vect}(\mathbb{R})$  of vector fields  $X_h = h \frac{d}{dx}$ , where  $h \in C^\infty(\mathbb{R})$ , acts by the *Lie derivative*. Alternatively, this action can be written as follows:

$$X_h \cdot (f dx^\mu) = L_{X_h}^\mu(f) dx^\mu \quad \text{with} \quad L_{X_h}^\mu(f) = hf' + \mu h'f, \quad (1.1)$$

where  $f', h'$  are  $\frac{df}{dx}, \frac{dh}{dx}$ . For  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$  and  $\mu \in \mathbb{R}$  we denote by  $D_{\lambda,\mu}$  the space of  $n$ -ary differential operators  $A$  from  $\mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n}$  to  $\mathcal{F}_\mu$ . The Lie algebra  $\text{Vect}(\mathbb{R})$  acts on the space  $D_{\lambda,\mu}$  of these differential operators by:

$$X_h \cdot A = L_{X_h}^\mu \circ A - A \circ L_{X_h}^\lambda \quad (1.2)$$

where  $L_{X_h}^\lambda$  is the Lie derivative on  $\mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n}$  defined by the Leibnitz rule. If we restrict ourselves to the Lie algebra  $\mathfrak{aff}(1)$  which is isomorphic to the Lie subalgebra of  $\text{Vect}(\mathbb{R})$  spanned by

$$\{X_1, X_x\},$$

we have a family of infinite dimensional  $\mathfrak{aff}(1)$ -modules still denoted by  $D_{\lambda,\mu}$ .

---

\*Université de Sfax, Faculté des Sciences, Département de Mathématiques, Laboratoire d'Algèbre, Géométrie et Théorie Spectrale (AGTS) LR11ES53, BP 802, 3038 Sfax, Tunisie. E.mail: mabrouk.benammar@fss.rnu.tn

According to Nijenhuis-Richardson [10], the space  $H^1(\mathfrak{g}; \text{End}(V))$  classifies the infinitesimal deformations of a  $\mathfrak{g}$ -module  $V$  and the obstructions to integrability of a given infinitesimal deformation of  $V$  are elements of  $H^2(\mathfrak{g}; \text{End}(V))$ . While the spaces  $H^1(\mathfrak{g}; L(\otimes_s^k V, V))$  appear naturally in the problem of normalization of nonlinear representations of  $\mathfrak{g}$  in  $V$ . To be more precise, let

$$T : \mathfrak{g} \rightarrow \sum_{k \geq 0} L(\otimes_s^k V, V), \quad X \mapsto T_X = \sum T_X^k,$$

be a nonlinear representation of  $\mathfrak{g}$  in  $V$ , that is,  $T_{[X,Y]} = [T_X, T_Y]$ . In [2], it is proved that the representation  $T$  is normalized if  $T_X^k$  is in a supplementary of  $B^1(\mathfrak{g}; L(\otimes_s^k V, V))$  in  $Z^1(\mathfrak{g}; L(\otimes_s^k V, V))$ .

In fact if  $A$  is a differential operator on the line,  $A$  can be viewed as an homomorphism from  $\mathcal{F}_\lambda$  to  $\mathcal{F}_\mu$ . If  $A$  is with order  $n$ , we can define its symbol as an element in  $\mathcal{S}_\delta^n = \bigoplus_{j=0}^n \mathcal{F}_{\delta-j}$  for  $\delta = \mu - \lambda$ . If  $n$  goes to  $+\infty$ , the space  $\mathcal{S}_\delta = \bigoplus_{j \geq 0} \mathcal{F}_{\delta-j}$  appears as the space of symbols for all differential operators. The space  $H^1(\mathfrak{g}; L(\otimes_s^2 \mathcal{S}_\delta, \mathcal{S}_\delta))$  can be decomposed as a sum of spaces  $H^1(\mathfrak{g}, D_{\lambda, \mu})$  with  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ . Thus, the computation of the spaces  $H^1(\mathfrak{g}, D_{\lambda, \mu})$  is the first step to normalize any nonlinear representation of  $\mathfrak{g}$  in  $\mathcal{S}_\delta$ .

For  $\lambda \in \mathbb{R}$  the spaces  $H_{\text{diff}}^1(\mathfrak{sl}(2), D_{\lambda, \mu})$  are computed by Gargoubi [7] and Lecomte [9] and the spaces  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu})$  are computed by Basdouri and Ben Ammar [3], where  $H_{\text{diff}}^1$  denotes the differential cohomology; that is, only cochains given by differential operators are considered. For  $\lambda \in \mathbb{R}^2$  the spaces  $H_{\text{diff}}^1(\mathfrak{sl}(2), D_{\lambda, \mu})$  are computed by Bouarroudj [5] and the spaces  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu})$  are computed by Ben Ammar et al [4], while we are interested in this paper in the spaces  $H_{\text{diff}}^1(\mathfrak{aff}(1), D_{\lambda, \mu})$  and  $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$  where  $\lambda \in \mathbb{R}^n$ .

## 2 Definitions and Notations

### 2.1 The Lie superalgebra of contact vector fields on $\mathbb{R}^{1|1}$

We define the superspace  $\mathbb{R}^{1|1}$  in terms of its superalgebra of functions, denoted by  $C^\infty(\mathbb{R}^{1|1})$  and consisting of elements of the form:

$$F(x, \theta) = f_0(x) + f_1(x)\theta,$$

where  $x$  is the even variable,  $\theta$  is the odd variable ( $\theta^2 = 0$ ) and  $f_0(x), f_1(x) \in C^\infty(\mathbb{R})$ . Even elements in  $C^\infty(\mathbb{R}^{1|1})$  are the functions  $F(x, \theta) = f_0(x)$ , the functions  $F(x, \theta) = \theta f_1(x)$  are odd elements. We consider the contact bracket on  $C^\infty(\mathbb{R}^{1|1})$  defined on  $C^\infty(\mathbb{R}^{1|1})$  by:

$$\{F, G\} = FG' - F'G + \frac{1}{2}D(F)\overline{D}(G),$$

where  $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$  and  $\overline{D} = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial x}$ . The superspace  $\mathbb{R}^{1|1}$  is equipped with the standard contact structure given by the following 1-form:

$$\alpha = dx + \theta d\theta.$$

Let  $\text{Vect}(\mathbb{R}^{1|1})$  be the superspace of vector fields on  $\mathbb{R}^{1|1}$ :

$$\text{Vect}(\mathbb{R}^{1|1}) = \left\{ F_0 \partial_x + F_1 \partial_\theta \mid F_i \in C^\infty(\mathbb{R}^{1|1}) \right\},$$

where  $\partial_\theta$  stands for  $\frac{\partial}{\partial\theta}$  and  $\partial_x$  stands for  $\frac{\partial}{\partial x}$ , and consider the superspace  $\mathcal{K}(1)$  of contact vector fields on  $\mathbb{R}^{1|1}$ . That is,  $\mathcal{K}(1)$  is the superspace of vector fields on  $\mathbb{R}^{1|1}$  preserving the distribution  $\langle \overline{D} \rangle$ :

$$\mathcal{K}(1) = \{X \in \text{Vect}(\mathbb{R}^{1|1}) \mid [X, \overline{D}] = F_X \overline{D} \text{ for some } F_X \in C^\infty(\mathbb{R}^{1|1})\}.$$

Any contact vector field on  $\mathbb{R}^{1|1}$  has the following explicit form:

$$X_F = F\partial_x + \frac{1}{2}D(F)\overline{D}, \text{ where } F \in C^\infty(\mathbb{R}^{1|1}).$$

The bracket on  $\mathcal{K}(1)$  is given by

$$[X_F, X_G] = X_{\{F, G\}}.$$

Thus, the map  $F \mapsto X_F$  is a Lie superalgebra isomorphism from  $\mathcal{K}(1)$  to  $C^\infty(\mathbb{R}^{1|1})$ . Of course, the set  $\{X_f = f\partial_x + \frac{1}{2}f'\theta\partial\theta, f \in C^\infty(\mathbb{R})\}$  is a subalgebra of  $\mathcal{K}(1)$  isomorphic to  $\text{Vect}(\mathbb{R})$ .

## 2.2 The subalgebra $\mathfrak{aff}(1|1)$

The Lie algebra  $\mathfrak{aff}(1)$  is realized as subalgebra of the Lie algebra  $\text{Vect}(\mathbb{R})$ :

$$\mathfrak{aff}(1) = \text{Span}(X_1, X_x).$$

Similarly, we now consider the Lie superalgebra  $\mathfrak{aff}(1|1)$  as a subalgebra of  $\mathcal{K}(1)$ :

$$\mathfrak{aff}(1|1) = \text{Span}(X_1, X_x, X_\theta).$$

The space of even elements is isomorphic to  $\mathfrak{aff}(1)$ , while the space of odd elements is one dimensional:

$$(\mathfrak{aff}(1|1))_{\bar{1}} = \text{Span}(X_\theta).$$

The commutation relations are

$$[X_x, X_\theta] = -\frac{1}{2}X_\theta, \quad [X_x, X_1] = -X_1, \quad [X_\theta, X_\theta] = \frac{1}{2}X_1.$$

## 2.3 The space of weighted densities on $\mathbb{R}^{1|1}$

We have analogous definition of weighted densities in super setting (see [1]) with  $dx$  replaced by  $\alpha$ . The elements of these spaces are indeed (weighted) densities since all spaces of generalized tensor fields have just one parameter relative  $\mathcal{K}(1)$  — the value of  $X_x$  on the lowest weight vector (the one annihilated by  $X_\theta$ ). From this point of view the volume element (roughly speaking, “ $dx \frac{\partial}{\partial\theta}$ ”) is indistinguishable from  $\alpha^{\frac{1}{2}}$ . We denote by  $\mathfrak{F}_\mu$  the space of all weighted densities on  $\mathbb{R}^{1|1}$  of weight  $\mu \in \mathbb{R}$ :

$$\mathfrak{F}_\mu = \left\{ F(x, \theta) \alpha^\mu \mid F(x, \theta) \in C^\infty(\mathbb{R}^{1|1}) \right\}.$$

As a vector space,  $\mathfrak{F}_\mu$  is isomorphic to  $C^\infty(\mathbb{R}^{1|1})$ , but the Lie derivative of the density  $G\alpha^\mu$  along the vector field  $X_F$  in  $\mathcal{K}(1)$  is now:

$$\mathfrak{L}_{X_F}(G\alpha^\mu) = \mathfrak{L}_{X_F}^\mu(G)\alpha^\mu, \quad \text{with } \mathfrak{L}_{X_F}^\mu(G) = \mathfrak{L}_{X_F}(G) + \mu F'G.$$

Especially, if  $f \in C^\infty(\mathbb{R})$  and  $G(x, \theta) = g_0(x) + g_1(x)\theta$ , then we easily check that

$$\mathfrak{L}_{X_f}^\mu(G) = L_{X_f}^\mu(g_0) + \left( L_{X_f}^{\mu+\frac{1}{2}}(g_1) \right) \theta. \quad (2.1)$$

Of course, for all  $\mu$ ,  $\mathfrak{F}_\mu$  is a  $\mathcal{K}(1)$ -module:

$$[\mathfrak{L}_{X_F}^\mu, \mathfrak{L}_{X_G}^\mu] = \mathfrak{L}_{[X_F, X_G]}^\mu.$$

## 2.4 Differential operators on weighted densities

A differential operator on  $\mathbb{R}^{1|1}$  is an operator on  $C^\infty(\mathbb{R}^{1|1})$  of the following form:

$$A = \sum_{i=0}^{\ell} a_i(x, \theta) \partial_x^i + \sum_{i=0}^{\ell} b_i(x, \theta) \partial_x^i \partial_\theta.$$

In [8], it is proved that any local operator  $A$  on  $\mathbb{R}^{1|1}$  is in fact a differential operator.

Of course, any differential operator defines a linear mapping from  $\mathfrak{F}_\lambda$  to  $\mathfrak{F}_\mu$  for any  $\lambda, \mu \in \mathbb{R}$ :  $F\alpha^\lambda \mapsto A(F)\alpha^\mu$ . Similarly, if  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$  and  $\mu \in \mathbb{R}$ , we consider a family of  $\mathcal{K}(1)$  and  $\mathfrak{aff}(1|1)$  modules denoted  $\mathfrak{D}_{\lambda, \mu}$ , for the natural action:

$$X_F \cdot A = \mathfrak{L}_{X_F}^\mu \circ A - (-1)^{AF} A \circ \mathfrak{L}_{X_F}^\lambda$$

where  $\mathfrak{L}_{X_F}^\lambda$  is the Lie derivative on  $\mathfrak{F}_{\lambda_1} \otimes \dots \otimes \mathfrak{F}_{\lambda_n}$  defined by the Leibnitz rule.

## 3 The space $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$

### 3.1 Cohomology

We will compute the first cohomology space of  $\mathfrak{aff}(1|1)$  with coefficients in  $\mathfrak{D}_{\lambda, \mu}$  where  $\lambda \in \mathbb{R}^n$  and  $\mu \in \mathbb{R}$ . Let us first recall some fundamental concepts from cohomology theory (see, e.g., [6]). Let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be a Lie superalgebra acting on a superspace  $V = V_0 \oplus V_1$  and let  $\mathfrak{h}$  be a subalgebra of  $\mathfrak{g}$ . (If  $\mathfrak{h}$  is omitted it assumed to be  $\{0\}$ ). The space of  $\mathfrak{h}$ -relative  $n$ -cochains of  $\mathfrak{g}$  with values in  $V$  is the  $\mathfrak{g}$ -module

$$C^n(\mathfrak{g}, \mathfrak{h}; V) := \text{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h}); V).$$

The *coboundary operator*  $\partial^n : C^n(\mathfrak{g}, \mathfrak{h}; V) \rightarrow C^{n+1}(\mathfrak{g}, \mathfrak{h}; V)$  is a  $\mathfrak{g}$ -map satisfying  $\partial^n \circ \partial^{n-1} = 0$ . The kernel of  $\partial^n$ , denoted  $Z^n(\mathfrak{g}, \mathfrak{h}; V)$ , is the space of  $\mathfrak{h}$ -relative  $n$ -cocycles, among them, the elements in the range of  $\partial^{n-1}$  are called  $\mathfrak{h}$ -relative  $n$ -coboundaries. We denote  $B^n(\mathfrak{g}, \mathfrak{h}; V)$  the space of  $n$ -coboundaries.

By definition, the  $n^{\text{th}}$   $\mathfrak{h}$ -relative cohomology space is the quotient space

$$H^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V) / B^n(\mathfrak{g}, \mathfrak{h}; V).$$

We will only need the formula of  $\partial^n$  (which will be simply denoted  $\partial$ ) in degrees 0 and 1. For  $v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{h}}$ ,

$$\partial v(g) := (-1)^{gv} g \cdot v,$$

where  $V^{\mathfrak{h}}$  is the subspace of  $\mathfrak{h}$ -invariant elements of  $V$ . For  $\Omega \in C^1(\mathfrak{g}, \mathfrak{h}; V)$  and  $g, h \in \mathfrak{g}$ ,

$$\partial(\Omega)(g, h) := (-1)^{g\Omega} g \cdot \Omega(h) - (-1)^{h(g+\Omega)} h \cdot \Omega(g) - \Upsilon([g, h]).$$

**Proposition 3.1.** 1) Let  $\Omega \in Z^1(\mathfrak{g}; V)$  and  $a \in \mathfrak{g}$ . If  $\Omega(a) = 0$  then  $a \cdot \Omega = 0$ . Thus, if  $\Omega|_{\mathfrak{h}} = 0$  then  $\Omega$  is  $\mathfrak{h}$ -invariant:  $\Omega \in Z^1(\mathfrak{g}, \mathfrak{h}; V)$ . Moreover,  $H^1(\mathfrak{g}, \mathfrak{h}; V) \subset H^1(\mathfrak{g}; V)$ .

2) If  $H^1(\mathfrak{g}, \mathfrak{h}; V) = 0$  then  $\Omega \in B^1(\mathfrak{g}; V)$  if and only if  $\Omega|_{\mathfrak{h}} \in B^1(\mathfrak{h}; V)$ . Thus, in this case, the space  $H^1(\mathfrak{g}; V)$  is characterized by the space  $H^1(\mathfrak{h}; V)$ .

Proof. 1) Indeed, for any  $x \in \mathfrak{g}$ , we have

$$(-1)^{a\Omega} a \cdot \Omega(x) - (-1)^{x(a+\Omega)} x \cdot \Omega(a) - \Omega([a, x]) = 0 \text{ and } (a \cdot \Omega)(x) = a \cdot \Omega(x) - (-1)^{a\Omega} \Omega([a, x]).$$

Moreover, for  $\Omega \in Z^1(\mathfrak{g}, \mathfrak{h}; V)$  we denote by  $\bar{\Omega}$  (respectively  $\tilde{\Omega}$ ) the class of  $\Omega$  up to  $B^1(\mathfrak{g}, \mathfrak{h}; V)$  (respectively  $B^1(\mathfrak{g}; V)$ ). We easily check that the map  $\bar{\Omega} \mapsto \tilde{\Omega}$ , from  $H^1(\mathfrak{g}, \mathfrak{h}; V)$  to  $H^1(\mathfrak{g}; V)$ , is injective : if  $\Omega = \partial v$  with  $v \in V$  then  $v \in V^{\mathfrak{h}}$  since  $\Omega|_{\mathfrak{h}} = 0$ .

2) Obviously if  $\Omega \in B^1(\mathfrak{g}; V)$  then  $\Omega|_{\mathfrak{h}} \in B^1(\mathfrak{h}; V)$ . Inversely, if  $\Omega|_{\mathfrak{h}} \in B^1(\mathfrak{h}; V)$  then, up to a coboundary,  $\Omega$  vanishes on  $\mathfrak{h}$ , therefore,  $\Omega \in Z^1(\mathfrak{g}, \mathfrak{h}; V)$ . But  $H^1(\mathfrak{g}, \mathfrak{h}; V) = \{0\} \subset H^1(\mathfrak{g}; V)$ , so,  $\Omega \in B^1(\mathfrak{g}; V)$ .  $\square$

### 3.2 The space $H_{\text{diff}}^1(\mathfrak{aff}(1), D_{\lambda, \mu})$

Let  $\mu \in \mathbb{R}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  and  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ , we consider  $\delta = \mu - \sum_{i=1}^n \lambda_i$  and  $|\alpha| = \sum \alpha_i$ . For  $F = f_1 \otimes \dots \otimes f_n \in \mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n}$ , we define

$$F^{(\alpha)} := f_1^{(\alpha_1)} \dots f_n^{(\alpha_n)}.$$

Recall that the space  $\mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n}$  is an  $\mathfrak{aff}(1)$ -module:

$$X_h \cdot F := L_{X_h}^\lambda(F) = \sum_{i=1}^n f_1 \otimes \dots \otimes L_{X_h}^{\lambda_i}(f_i) \otimes \dots \otimes f_n.$$

The following lemma gives the general form of any 1-cocycle.

**Lemma 3.1.** Up to a coboundary, any 1-cocycle  $\Omega \in Z_{\text{diff}}^1(\mathfrak{aff}(1), D_{\lambda, \mu})$  can be expressed as follows. For all  $F = f_1 \otimes \dots \otimes f_n \in \mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n}$  and for all  $X_h \in \mathfrak{aff}(1)$ :

$$\Omega(X_h, F) = \sum_{\alpha} B_{\alpha} h' F^{(\alpha)}, \quad (3.1)$$

where the  $B_{\alpha}$  are constants.

Proof. Any 1-cocycle on  $\mathfrak{aff}(1)$  should retain the following general form:

$$\Omega(X_h, F) = \sum_{\alpha} N_{\alpha} h F^{(\alpha)} + \sum_{\alpha} M_{\alpha} h' F^{(\alpha)},$$

where  $N_{\alpha}$  and  $M_{\alpha}$  are, a priori, functions. First, we prove that the terms in  $h$  can be annihilated by adding a coboundary. Let  $b : \mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n} \rightarrow \mathcal{F}_{\mu}$  be a  $n$ -ary differential operator defined by

$$b(F) = \sum_{\alpha} D_{\alpha} F^{(\alpha)},$$

We have

$$\begin{aligned}\partial b(X_h, F) &= h(b(F))' + \mu h' b(F) - b(X_h \cdot F) \\ &= \sum_{\alpha} D'_{\alpha} h F^{(\alpha)} + \sum_{\alpha} (\delta - |\alpha|) D_{\alpha} h' F^{(\alpha)}\end{aligned}\tag{3.2}$$

Thus, if  $D'_{\alpha} = N_{\alpha}$  then  $\Omega - \partial b$  does not contain terms in  $h$ . So, we can replace  $\Omega$  by  $\Omega - \partial b$ . That is, up to a coboundary, any 1-cocycle on  $\mathfrak{aff}(1)$  can be expressed as follows:

$$\Omega(X_h, F) = \sum_{\alpha} B_{\alpha} h' F^{(\alpha)}.$$

Now, consider the 1-cocycle condition:

$$\Omega([X_{h_1}, X_{h_2}], F) - X_{h_1} \cdot \Omega(X_{h_2}, F) + X_{h_2} \cdot \Omega(X_{h_1}, F) = 0,$$

where  $X_{h_1}, X_{h_2} \in \mathfrak{aff}(1)$ . That is,

$$\sum_{\alpha} B'_{\alpha} (h_1 h_2' - h_1' h_2) F^{(\alpha)} = 0.$$

So, for all  $\alpha$ , we have  $B'_{\alpha} = 0$  □

**Theorem 3.2.** 1) If  $\delta \notin \mathbb{N}$  then  $H^1_{\text{diff}}(\mathfrak{aff}(1), D_{\lambda, \mu}) = 0$ .

2) If  $\delta \in \mathbb{N}$  then, up to a coboundary, any 1-cocycle  $c \in Z^1_{\text{diff}}(\mathfrak{aff}(1), D_{\lambda, \mu})$  can be expressed as follows. For all  $F = f_1 \otimes \cdots \otimes f_n \in \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$  and for all  $X_h \in \mathfrak{aff}(1)$ :

$$\Omega(X_h, F) = \sum_{|\alpha|=\delta} B_{\alpha} h' F^{(\alpha)},\tag{3.3}$$

Proof. 1) Indeed, according to Lemma 3.1, we can easily show the 1-cocycle  $\Omega$  defined by (3.1) is nothing but the operator  $\partial b$  where

$$b(F) = \sum_{\alpha} \frac{B_{\alpha}}{\delta - |\alpha|} F^{(\alpha)},$$

2) Consider the 1-cocycle  $\Omega$  defined by (3.1) and consider the operator  $\partial b$  where

$$b(F) = \sum_{|\alpha| \neq \delta} \frac{B_{\alpha}}{\delta - |\alpha|} F^{(\alpha)}.$$

We easily show that

$$(\Omega - \partial b)(X_h, F) = \sum_{|\alpha|=\delta} B_{\alpha} h' F^{(\alpha)}.$$

□

**Theorem 3.3.** If  $\delta = k \in \mathbb{N}$  then

$$\dim H^1(\mathfrak{aff}(1); D_{\lambda, \mu}) = \binom{n+k-1}{k}.$$

Proof. According to Theorem 3.2 and to the formula (3.2), the classes of 1-cocycles  $\Omega^{\alpha}$  defined by  $\Omega^{\alpha}(X_h, F) = h' F^{(\alpha)}$ , where  $|\alpha| = k$ , constitute a basis of  $H^1(\mathfrak{aff}(1); D_{\lambda, \mu})$ . Thus,  $\dim H^1(\mathfrak{aff}(1); D_{\lambda, \mu})$  is the cardinal of the set  $\{\alpha \in \mathbb{N}^n, |\alpha| = k\}$ .

## 4 Relationship between $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$ and $H_{\text{diff}}^1(\mathfrak{aff}(1), D_{\lambda, \mu})$

We need to present here some results illustrating the analogy between the cohomology spaces in super and classical settings. We consider

$$\begin{aligned}\mathcal{E} &= \left\{ \varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \quad \varepsilon_i = 0, \frac{1}{2}, \quad \text{and} \quad |\varepsilon| \in \mathbb{N} \right\}, \\ \mathcal{O} &= \left\{ \varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \quad \varepsilon_i = 0, \frac{1}{2}, \quad \text{and} \quad |\varepsilon| \notin \mathbb{N} \right\}.\end{aligned}$$

### Proposition 4.1.

1) As a  $\mathfrak{aff}(1)$ -module, we have

$$\mathfrak{F}_\mu \simeq \mathcal{F}_\mu \oplus \Pi(\mathcal{F}_{\mu + \frac{1}{2}}) \quad \text{and} \quad \mathfrak{aff}(1|1) \simeq \mathfrak{aff}(1) \oplus \Pi(\mathbb{R}dx^{-\frac{1}{2}}),$$

where  $\mu \in \mathbb{R}$  and  $\Pi$  is the change of parity.

2) For  $\lambda \in \mathbb{R}^n$  and  $\mu \in \mathbb{R}$ , as a  $\mathfrak{aff}(1)$ -module, we have, for the homogeneous relative parity components:

$$(\mathfrak{D}_{\lambda, \mu})_{\bar{0}} \simeq \bigoplus_{\varepsilon \in \mathcal{E}} D_{(\lambda + \varepsilon, \mu)} \oplus \bigoplus_{\varepsilon \in \mathcal{O}} D_{(\lambda + \varepsilon, \mu + \frac{1}{2})} \quad \text{and} \quad (\mathfrak{D}_{\lambda, \mu})_{\bar{1}} \simeq \Pi \left( \bigoplus_{\varepsilon \in \mathcal{O}} D_{(\lambda + \varepsilon, \mu)} \oplus \bigoplus_{\varepsilon \in \mathcal{E}} D_{(\lambda + \varepsilon, \mu + \frac{1}{2})} \right). \quad (4.1)$$

Proof. The first statement is immediately deduced from (2.1) and from the fact that  $\mathcal{K}(1) \simeq \mathfrak{F}_{-\frac{1}{2}}$ . The second statement can be deduced from the first one (see [4]).  $\square$

Now, in order to describe  $H^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$ , we need first to describe the  $\mathfrak{aff}(1)$ -relative cohomology space  $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{aff}(1); \mathfrak{D}_{\lambda, \mu})$ . So, we shall need the following description of some  $\mathfrak{aff}(1)$ -invariant mappings.

### Lemma 4.1. Let

$$\begin{aligned}A : \mathbb{R}dx^{-\frac{1}{2}} \otimes \mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n} &\longrightarrow \mathcal{F}_\mu, \\ (adx^{-\frac{1}{2}}, f_1 dx^{\lambda_1}, \dots, f_n dx^{\lambda_n}) &\longmapsto A(a, f_1, \dots, f_n)(dx)^\mu\end{aligned} \quad (4.2)$$

be an  $n+1$ -ary differential operator and let  $\delta = \mu - \sum_i \lambda_i$ . If  $A$  is a nontrivial  $\mathfrak{aff}(1)$ -invariant operator then

$$\delta + \frac{1}{2} \in \mathbb{N}.$$

The corresponding operator  $A$  is of the form:

$$A(a, f_1, \dots, f_n) = \sum_{|\alpha| = \delta + \frac{1}{2}} c_\alpha F^{(\alpha)},$$

where the  $c_\alpha$  are constants.

Proof. The invariance with respect the vector field  $X_1 = \partial_x$  yields that  $A$  must be expressed with constant coefficients. Thus, the operator  $A$  can be expressed as follows:

$$A(a, f_1, \dots, f_n) = \sum_{\alpha} c_\alpha F^{(\alpha)},$$

where the  $c_\alpha$  are constants,  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  and  $F^{(\alpha)} = f_1^{(\alpha_1)} \dots f_n^{(\alpha_n)}$ . Note that the operator  $A$  can be viewed as element of  $D_{\lambda, \mu + \frac{1}{2}}$ .

The invariance property of  $A$  with respect the vector fields  $X_x$  reads:

$$\begin{aligned} 0 = & x(A(f_1, \dots, f_n))' + (\mu + \frac{1}{2})A(f_1, \dots, f_n) - A(xf'_1 + \lambda_1 f_1, f_2, \dots, f_n) \\ & - A(f_1, xf'_2 + \lambda_2 f_2, f_3, \dots, f_n) - \dots - A(f_1, f_2, \dots, xf'_n + \lambda_n f_n). \end{aligned} \quad (4.3)$$

Consider any non vanishing coefficient  $c_\alpha$  and consider terms in  $F^{(\alpha)}$  in (4.3), we get

$$\delta + \frac{1}{2} = |\alpha|.$$

□

**Proposition 4.2.** *The  $\mathfrak{aff}(1)$ -relative cohomology spaces  $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{aff}(1); \mathfrak{D}_{\lambda, \mu})$  are all trivial. That is, any 1-cocycle  $\Omega$  is a coboundary over  $\mathfrak{aff}(1|1)$  if and only if its restriction to  $\mathfrak{aff}(1)$  is a coboundary over  $\mathfrak{aff}(1)$ .*

Proof. First, it is well known that the space  $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{aff}(1); \mathfrak{D}_{\lambda, \mu})$  is nothing but the space of cohomology classes of 1-cocycles vanishing on  $\mathfrak{aff}(1)$  [4].

Let  $\Omega$  be a 1-cocycle vanishing on  $\mathfrak{aff}(1)$ , then, by the 1-cocycle condition, we have:

$$X_1 \cdot \Omega(X_\theta) - \Omega([X_1, X_\theta]) = 0, \quad (4.4)$$

$$X_x \cdot \Omega(X_\theta) - \Omega([X_x, X_\theta]) = 0, \quad (4.5)$$

$$X_\theta \cdot \Omega(X_\theta) = 0. \quad (4.6)$$

The equations (4.4) and (4.5) express the  $\mathfrak{aff}(1)$ -invariance property of the map  $\Omega$ . According to the decomposition (4.1) the map  $\Omega(X_\theta)$  is decomposed into some  $\mathfrak{aff}(1)$ -invariant elements

$\Omega$  of  $\bigoplus_{\varepsilon \in \mathcal{E}} D_{(\lambda+\varepsilon, \mu)} \oplus \bigoplus_{\varepsilon \in \mathcal{O}} D_{(\lambda+\varepsilon, \mu+\frac{1}{2})}$  or  $\Pi \left( \bigoplus_{\varepsilon \in \mathcal{O}} D_{(\lambda+\varepsilon, \mu)} \oplus \bigoplus_{\varepsilon \in \mathcal{E}} D_{(\lambda+\varepsilon, \mu+\frac{1}{2})} \right)$ , according to that  $\Omega$  is odd or even. Therefore, the expressions of these maps are given by Lemma 4.1, in fact, the change of parity functor  $\Pi$  commutes with the the  $\mathfrak{aff}(1)$ -action.

So, if  $\Omega$  is an even 1-cocycle then we must have  $\delta \in \mathbb{N}$  and if  $\Omega$  is an odd 1-cocycle then we must have  $\delta + \frac{1}{2} \in \mathbb{N}$ . Otherwise, the operator  $\Omega$  is identically the zero map.

If  $\Omega$  is an even 1-cocycle then  $\Omega(X_\theta)$  is odd, so

$$\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, |\alpha|=\delta_\varepsilon+\frac{1}{2}} C_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha|=\delta_\varepsilon+1} D_\alpha^\varepsilon \Omega_\varepsilon^\alpha,$$

where, if  $F_i = f_i + \theta g_i$ , then

$$\Omega_\varepsilon^\alpha(F_1, \dots, F_n) = H_\varepsilon^{(\alpha)}$$

with

$$H_\varepsilon^{(\alpha)} = h_1^{(\alpha_1)} \dots h_n^{(\alpha_n)} \quad \text{where} \quad h_i = \begin{cases} f_i & \text{if } \varepsilon_i = 0 \\ g_i & \text{if } \varepsilon_i = \frac{1}{2} \end{cases}$$

**Lemma 4.2.** *For any  $X_h \in \mathfrak{aff}(1)$  we have*

$$\partial \theta \Omega_\varepsilon^\alpha(X_h) = (\delta_\varepsilon + \frac{1}{2} - |\alpha|) \theta \Omega_\varepsilon^\alpha,$$

where  $\Omega_\varepsilon^\alpha$  is viewed as element of  $\mathfrak{D}_{\lambda, \mu}$ .



Proof. By a straightforward computation. We can also see  $\Omega_\varepsilon^\alpha$  as element of  $D_{\lambda+\varepsilon, \mu+\frac{1}{2}}$ , so, we deduce the result from (3.2).

Now, since  $X_\theta = \frac{1}{2}D$ , then by the equation (4.6) we have

$$D(\Omega(X_\theta, F_1, \dots, F_n)) = -\sum_{i=1}^n \xi_i \Omega(X_\theta, F_1, \dots, D(F_i), \dots, F_n) \quad (4.7)$$

where

$$\xi_i = (-1)^{\sum_{j=1}^{i-1} F_j}.$$

For  $\varepsilon \in \mathcal{E} \cup \mathcal{O}$  and  $i \in \{1, \dots, n\}$  we consider the element  $\varepsilon^i = \eta \in \mathcal{E} \cup \mathcal{O}$  defined by

$$\eta_i \neq \varepsilon_i \quad \text{and} \quad \eta_j = \varepsilon_j \quad \text{for} \quad j \neq i$$

and we define  $\xi_\varepsilon^i$  by

$$\xi_\varepsilon^i = (-1)^{2 \sum_{j < i} \varepsilon_j}.$$

Note that

$$\xi_\varepsilon^i = \xi_{\varepsilon^i}^i. \quad (4.8)$$

For  $\alpha \in \mathbb{N}^n$ , we define

$$\alpha^i = (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_n) \quad \text{and} \quad \alpha^{\bar{i}} = (\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_n).$$

Note that

$$(\varepsilon^i)^i = \varepsilon, \quad (\alpha^{\bar{i}})^i = (\alpha^i)^{\bar{i}} = \alpha \quad \text{and} \quad \varepsilon \in \mathcal{E} \Leftrightarrow \varepsilon^i \in \mathcal{O}. \quad (4.9)$$

If  $F = f + \theta g$  then  $D(F) = g + \theta f'$ . Thus,

$$\Omega_\varepsilon^\alpha(F_1, \dots, D(F_i), \dots, F_n) = \Omega_{\varepsilon^i}^{\alpha^i}(F_1, \dots, \dots, F_n).$$

where

$$\alpha_\varepsilon^i = \begin{cases} \alpha & \text{if } \varepsilon_i = 0 \\ \alpha^i & \text{if } \varepsilon_i = \frac{1}{2} \end{cases}.$$

Define  $\alpha_\varepsilon^{\bar{i}}$  so that

$$(\alpha_\varepsilon^{\bar{i}})^i = \alpha, \quad (4.10)$$

that is

$$\alpha_\varepsilon^{\bar{i}} = \begin{cases} \alpha & \text{if } \varepsilon_i = \frac{1}{2} \\ \alpha^{\bar{i}} & \text{if } \varepsilon_i = 0 \end{cases}.$$

Thus, the equation (4.7) becomes

$$\sum_{\varepsilon \in \mathcal{E}, \alpha} D_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} C_\alpha^\varepsilon \sum_{i=1}^n \Omega_\varepsilon^{\alpha^i} = - \sum_{\varepsilon \in \mathcal{O}, \alpha} C_\alpha^\varepsilon \sum_{i=1}^n \xi_\varepsilon^i \Omega_{\varepsilon^i}^{\alpha_\varepsilon^i} - \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} D_\alpha^\varepsilon \sum_{i=1}^n \xi_\varepsilon^i \Omega_{\varepsilon^i}^{\alpha_\varepsilon^i} \quad (4.11)$$

or equivalently (according to (4.9), (4.8) and (4.10))

$$\sum_{\varepsilon \in \mathcal{E}, \alpha} D_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( \sum_{i=1}^n C_{\alpha^{\bar{i}}}^\varepsilon \right) \Omega_\varepsilon^\alpha = - \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^n \xi_\varepsilon^i C_{\alpha_\varepsilon^i}^\varepsilon \right) \Omega_\varepsilon^\alpha - \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( \sum_{i=1}^n \xi_\varepsilon^i D_{\alpha_\varepsilon^i}^\varepsilon \right) \Omega_\varepsilon^\alpha \quad (4.12)$$

Thus, the coefficients  $C_\alpha^\varepsilon$  must satisfy the following conditions

$$\begin{cases} D_\alpha^\varepsilon &= -\sum_{i=1}^n \xi_\varepsilon^i C_{\alpha_{\varepsilon^i}^\varepsilon}^\varepsilon & \text{for } \varepsilon \in \mathcal{E} \\ \sum_{i=1}^n C_{\alpha^i}^\varepsilon &= -\sum_{i=1}^n \xi_\varepsilon^i D_{\alpha_{\varepsilon^i}^\varepsilon}^\varepsilon & \text{for } \varepsilon \in \mathcal{O}. \end{cases} \quad (4.13)$$

Thus,

$$\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, \alpha} C_\alpha^\varepsilon \Omega_\varepsilon^\alpha - \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^n \xi_\varepsilon^i C_{\alpha_{\varepsilon^i}^\varepsilon}^\varepsilon \right) \Omega_\varepsilon^\alpha = \sum_{\varepsilon \in \mathcal{O}, \alpha} C_\alpha^\varepsilon \Omega_\varepsilon^\alpha - \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} C_\alpha^\varepsilon \sum_{i=1}^n \xi_\varepsilon^i \Omega_{\varepsilon^i}^{\alpha_{\varepsilon^i}^\varepsilon}$$

with

$$\sum_{i=1}^n C_{\alpha^i}^\varepsilon = \sum_{i,j} \xi_\varepsilon^i \xi_{\varepsilon^i}^j C_{\alpha_{\varepsilon^i}^{\varepsilon^j}}^{\varepsilon^i}, \quad (4.14)$$

where  $\alpha_{\varepsilon^i}^{\varepsilon^j} = (\alpha_{\varepsilon^i}^\varepsilon)^{\varepsilon^j}$ . But, it is easy to see that

$$\xi_\varepsilon^i \xi_{\varepsilon^i}^j = -\xi_\varepsilon^j \xi_{\varepsilon^j}^i, \quad \varepsilon^{ii} = \varepsilon \quad \text{and} \quad \alpha_{\varepsilon^i}^{\varepsilon^i} = \alpha^i.$$

Therefore the relation (4.14) is trivial. Of course, we agree that if  $\alpha_{\varepsilon^i}^\varepsilon \notin \mathbb{N}^n$  or  $\alpha^i \notin \mathbb{N}^n$  then the correspondent coefficients  $C_{\alpha_{\varepsilon^i}^\varepsilon}^{\varepsilon^i}$  or  $C_{\alpha^i}^\varepsilon$  are zero.

Now, let  $B \in (\mathfrak{D}_{\lambda, \mu})_{\bar{0}}$ , according to the decomposition (4.1), we can write

$$\begin{aligned} B &= \sum_{\varepsilon \in \mathcal{E}, |\alpha|=\delta_\varepsilon+1} B_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha|=\delta_\varepsilon+\frac{1}{2}} R_\alpha^\varepsilon \Omega_\varepsilon^\alpha. \\ \partial B(X_\theta) &= \frac{1}{2} \sum_{\varepsilon \in \mathcal{O}, |\alpha|=\delta_\varepsilon+\frac{1}{2}} \left( R_\alpha^\varepsilon - \sum_{i=1}^n \xi_\varepsilon^i B_{\alpha_{\varepsilon^i}^\varepsilon}^{\varepsilon^i} \right) \Omega_\varepsilon^\alpha + \frac{1}{2} \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha|=\delta_\varepsilon+1} \left( \sum_{i=1}^n B_{\alpha^i}^{\varepsilon^i} - \sum_{i=1}^n \xi_\varepsilon^i R_{\alpha_{\varepsilon^i}^\varepsilon}^{\varepsilon^i} \right) \Omega_\varepsilon^\alpha. \end{aligned} \quad (4.15)$$

Thus, according to Lemma 4.2, we have

$$\Omega = \partial B$$

where

$$B = 2\theta \sum_{\varepsilon \in \mathcal{O}, |\alpha|=\delta_\varepsilon+\frac{1}{2}} C_\alpha^\varepsilon \Omega_\varepsilon^\alpha.$$

Now, if  $\Omega$  is an odd 1-cocycle then  $\delta + \frac{1}{2} \in \mathbb{N}$  and  $\Omega(X_\theta)$  is even, so

$$\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{E}, |\alpha|=\delta_\varepsilon+\frac{1}{2}} C_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha|=\delta_\varepsilon+1} D_\alpha^\varepsilon \Omega_\varepsilon^\alpha.$$

By the equation (4.6) we have

$$D(\Omega(X_\theta, F_1, \dots, F_n)) = \sum_{i=1}^n \xi_i \Omega(X_\theta, F_1, \dots, D(F_i), \dots, F_n). \quad (4.16)$$

Thus, the equation (4.7) becomes

$$\sum_{\varepsilon \in \mathcal{O}, \alpha} D_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} C_\alpha^\varepsilon \sum_{i=1}^n \Omega_\varepsilon^{\alpha^i} = \sum_{\varepsilon \in \mathcal{E}, \alpha} C_\alpha^\varepsilon \sum_{i=1}^n \xi_\varepsilon^i \Omega_{\varepsilon^i}^{\alpha_{\varepsilon^i}^\varepsilon} + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} D_\alpha^\varepsilon \sum_{i=1}^n \xi_\varepsilon^i \Omega_{\varepsilon^i}^{\alpha_{\varepsilon^i}^\varepsilon} \quad (4.17)$$

or equivalently (according to (4.9) and (4.10))

$$\sum_{\varepsilon \in \mathcal{O}, \alpha} D_{\alpha}^{\varepsilon} \Omega_{\varepsilon}^{\alpha} + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^n C_{\alpha^i}^{\varepsilon} \right) \Omega_{\varepsilon}^{\alpha} = \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^n \xi_{\varepsilon}^i C_{\alpha^i}^{\varepsilon} \right) \Omega_{\varepsilon}^{\alpha} + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( \sum_{i=1}^n \xi_{\varepsilon}^i D_{\alpha^i}^{\varepsilon} \right) \Omega_{\varepsilon}^{\alpha} \quad (4.18)$$

Thus, the coefficients  $C_{\alpha}^{\varepsilon}$  must satisfy the following conditions

$$D_{\alpha}^{\varepsilon} = \sum_{i=1}^n \xi_{\varepsilon}^i C_{\alpha^i}^{\varepsilon} \quad \text{for } \varepsilon \in \mathcal{O}. \quad (4.19)$$

Thus,

$$\Omega(X_{\theta}) = \sum_{\varepsilon \in \mathcal{E}, \alpha} C_{\alpha}^{\varepsilon} \Omega_{\varepsilon}^{\alpha} + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( \sum_{i=1}^n \xi_{\varepsilon}^i C_{\alpha^i}^{\varepsilon} \right) \Omega_{\varepsilon}^{\alpha} = \sum_{\varepsilon \in \mathcal{E}, \alpha} C_{\alpha}^{\varepsilon} \Omega_{\varepsilon}^{\alpha} + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} C_{\alpha}^{\varepsilon} \sum_{i=1}^n \xi_{\varepsilon}^i \Omega_{\varepsilon^i}^{\alpha^i}.$$

Now, let  $B \in (\mathfrak{D}_{\lambda, \mu})_{\bar{1}}$ , according to the decomposition (4.1), we can write

$$B = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_{\varepsilon} + 1} B_{\alpha}^{\varepsilon} \Omega_{\varepsilon}^{\alpha} + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_{\varepsilon} + \frac{1}{2}} R_{\alpha}^{\varepsilon} \Omega_{\varepsilon}^{\alpha}.$$

$$\partial B(X_{\theta}) = \frac{1}{2} \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( R_{\alpha}^{\varepsilon} + \sum_{i=1}^n \xi_{\varepsilon}^i B_{\alpha^i}^{\varepsilon} \right) \Omega_{\varepsilon}^{\alpha} + \frac{1}{2} \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^n B_{\alpha^i}^{\varepsilon} + \sum_{i=1}^n \xi_{\varepsilon}^i R_{\alpha^i}^{\varepsilon} \right) \Omega_{\varepsilon}^{\alpha}. \quad (4.20)$$

Thus, according to Lemma 4.2, we have

$$\Omega = \partial B$$

where

$$B = 2\theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_{\varepsilon} + \frac{1}{2}} C_{\alpha}^{\varepsilon} \Omega_{\varepsilon}^{\alpha}.$$

□

**Corollary 4.3.** *Any 1-cocycle  $\Omega \in Z_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$  of  $\mathfrak{aff}(1|1)$  is a coboundary if and only if its restriction to  $\mathfrak{aff}(1)$  is a coboundary.*

## 5 The space $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$

According to Theorem 3.2 the restriction of any 1-cocycle  $\Omega \in Z_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$  to  $\mathfrak{aff}(1)$  has the following structure:

**Proposition 5.1.** *1) If  $2\delta \notin \mathbb{N}$  then  $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}) = 0$ .*

*2) If  $\delta \in \mathbb{N}$  then, up to a coboundary, any 1-cocycle  $\Omega \in Z_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$  is even and its restriction to  $\mathfrak{aff}(1)$  has the following form:*

$$\Omega(X_h) = \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_{\varepsilon}} C_{\alpha}^{\varepsilon} h' \Omega_{\varepsilon}^{\alpha} + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_{\varepsilon} + \frac{1}{2}} D_{\alpha}^{\varepsilon} h' \Omega_{\varepsilon}^{\alpha}$$

where  $\delta_{\varepsilon} = \mu - |\lambda + \varepsilon|$ .

3) If  $\delta \in \mathbb{N} + \frac{1}{2}$  then, up to a coboundary, any 1-cocycle  $\Omega \in Z_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu})$  is odd and its restriction to  $\mathfrak{aff}(1)$  has the following form:

$$\Omega(X_h) = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon} C_\alpha^\varepsilon h' \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} D_\alpha^\varepsilon h' \Omega_\varepsilon^\alpha.$$

**Theorem 5.1.** 1) If  $2\delta \notin \mathbb{N}$  then  $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}) = 0$ .

2) If  $\delta = k \in \mathbb{N}$  then

$$\dim H^1(\mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} \binom{n+k-r-1}{k-r}.$$

3) If  $\delta = k + \frac{1}{2} \in \mathbb{N} + \frac{1}{2}$  then

$$\dim H^1(\mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} \binom{n+k-r-1}{k-r}.$$

A basis of  $H^1(\mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu})$  is given by the family  $\Gamma_\varepsilon^\alpha$  defined on  $\mathfrak{aff}(1)$  by

$$\Gamma_\varepsilon^\alpha(X_h) = h' \Omega_\varepsilon^\alpha + \theta \sum_{i=1}^n \xi_\varepsilon^i h' \Omega_{\varepsilon^i}^{\alpha^i}$$

where  $\varepsilon \in \mathcal{E}$  and  $|\alpha| = \delta_\varepsilon$  if  $\delta \in \mathbb{N}$  and  $\varepsilon \in \mathcal{O}$  and  $|\alpha| = \delta_\varepsilon + \frac{1}{2}$  if  $\delta \in \mathbb{N} + \frac{1}{2}$ .

Proof. If  $\delta \in \mathbb{N}$  then, according to Proposition 5.1, we have for any  $X_h \in \mathfrak{aff}(1)$

$$\Omega(X_h) = \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon} C_\alpha^\varepsilon h' \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} D_\alpha^\varepsilon h' \Omega_\varepsilon^\alpha.$$

In this case,  $X_\theta$  is an odd operator, so, it is of the following form:

$$\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, \alpha} B_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} R_\alpha^\varepsilon \Omega_\varepsilon^\alpha.$$

Using the 1-cocycle conditions we seek to establish relationships that must be satisfied by the coefficients  $B_\alpha^\varepsilon, R_\alpha^\varepsilon, C_\alpha^\varepsilon, \dots$

The 1-cocycle condition reads:

$$X_x \cdot \Omega(X_\theta) - X_\theta \cdot \Omega(X_x) + \frac{1}{2} \Omega(X_\theta) = 0, \quad (5.1)$$

where

$$\begin{aligned} X_x \cdot \Omega(X_\theta) &= \mathfrak{L}_{X_x}^\mu \circ \Omega(X_\theta) - \Omega(X_\theta) \circ \mathfrak{L}_{X_x}^\lambda, \\ \Omega_\varepsilon^\alpha \circ \mathfrak{L}_{X_x}^\lambda &= \sum_{i=1}^n \left( x \Omega_\varepsilon^{\alpha^i} + (\alpha_i + \lambda_i + \varepsilon_i) \Omega_\varepsilon^\alpha \right), \\ \mathfrak{L}_{X_x}^\mu \circ \Omega(X_\theta) &= \sum_{\varepsilon \in \mathcal{O}, \alpha} B_\alpha^\varepsilon L_{X_x}^\mu \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} R_\alpha^\varepsilon L_{X_x}^{\mu + \frac{1}{2}} \Omega_\varepsilon^\alpha, \\ L_{X_x}^\mu \Omega_\varepsilon^\alpha &= x \sum_{i=1}^n \Omega_\varepsilon^{\alpha^i} + \mu \Omega_\varepsilon^\alpha, \\ 2X_\theta \cdot \Omega(X_x) &= \sum_{\varepsilon \in \mathcal{O}, \alpha} D_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} C_\alpha^\varepsilon \sum_{i=1}^n \Omega_\varepsilon^{\alpha^i} - \\ &\quad \sum_{\varepsilon \in \mathcal{E}, \alpha} C_\alpha^\varepsilon \sum_{i=1}^n \xi_\varepsilon^i \Omega_{\varepsilon^i}^{\alpha^i} - \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} D_\alpha^\varepsilon \sum_{i=1}^n \xi_\varepsilon^i \Omega_{\varepsilon^i}^{\alpha^i} \end{aligned}$$

Therefore, the relation (5.1) is satisfied if and only if

$$\begin{cases} (\delta_\varepsilon - |\alpha| + \frac{1}{2})B_\alpha^\varepsilon - \frac{1}{2}D_\alpha^\varepsilon + \frac{1}{2}\left(\sum_{i=1}^n \xi_\varepsilon^i C_{\alpha_\varepsilon^i}^{\varepsilon^i}\right) = 0 \\ (\delta_\varepsilon - |\alpha| + 1)R_\alpha^\varepsilon - \frac{1}{2}\sum_{i=1}^n C_{\alpha^i}^\varepsilon + \frac{1}{2}\sum_{i=1}^n \xi_\varepsilon^i D_{\alpha_\varepsilon^i}^{\varepsilon^i} = 0. \end{cases}$$

Thus, for  $|\alpha| = \delta_\varepsilon + \frac{1}{2}$  we have

$$D_\alpha^\varepsilon = \sum_{i=1}^n \xi_\varepsilon^i C_{\alpha_\varepsilon^i}^{\varepsilon^i}$$

and for  $|\alpha| \neq \delta_\varepsilon + \frac{1}{2}$  we have  $|\alpha_\varepsilon^i| \neq \delta_{\varepsilon^i}$ , therefore  $D_\alpha^\varepsilon = C_{\alpha_\varepsilon^i}^{\varepsilon^i} = 0$  (indeed  $D_\alpha^\varepsilon$  appear only for  $|\alpha| = \delta_\varepsilon + \frac{1}{2}$  and  $C_\alpha^\varepsilon$  appear only for  $|\alpha| = \delta_\varepsilon$ ). Therefore,  $B_\alpha^\varepsilon = 0$  for  $|\alpha| \neq \delta_\varepsilon + \frac{1}{2}$ .

For  $|\alpha| = \delta_\varepsilon + 1$  we have

$$\sum_{i=1}^n C_{\alpha^i}^\varepsilon = \sum_{i=1}^n \xi_\varepsilon^i D_{\alpha_\varepsilon^i}^{\varepsilon^i}$$

and for  $|\alpha| \neq \delta_\varepsilon + 1$  we have  $C_{\alpha^i}^\varepsilon = D_{\alpha_\varepsilon^i}^{\varepsilon^i} = 0$ , therefore  $R_\alpha^\varepsilon = 0$ .

Thus,

$$\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} B_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + 1} R_\alpha^\varepsilon \Omega_\varepsilon^\alpha$$

and

$$\Omega(X_h) = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon} C_\alpha^\varepsilon h' \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} \sum_{i=1}^n \xi_\varepsilon^i C_{\alpha_\varepsilon^i}^{\varepsilon^i} h' \Omega_\varepsilon^\alpha$$

From Corollary 4.3 that the dimension of  $H^1(\mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu})$  is equal to the number of parameters  $C_\alpha^\varepsilon$ . That is,

$$\dim H^1(\mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} \binom{n+k-r-1}{k-r},$$

where  $\binom{n}{2r}$  is the number of  $\varepsilon$  with  $|\varepsilon| = r$  and  $\binom{n+k-r-1}{k-r}$  is the number of  $\alpha$  with  $|\alpha| = \delta_\varepsilon = k - r$ .

Now, If  $\delta \in \mathbb{N} + \frac{1}{2}$  then we have

$$\Omega(X_h) = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon} C_\alpha^\varepsilon h' \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} D_\alpha^\varepsilon h' \Omega_\varepsilon^\alpha.$$

In this case,  $X_\theta$  is an even operator, so, it is of the following form:

$$\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{E}, \alpha} B_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} R_\alpha^\varepsilon \Omega_\varepsilon^\alpha.$$

As before we prove that

$$\begin{cases} (\delta_\varepsilon - |\alpha| + \frac{1}{2})B_\alpha^\varepsilon - \frac{1}{2}D_\alpha^\varepsilon + \frac{1}{2}\left(\sum_{i=1}^n \xi_\varepsilon^i C_{\alpha_\varepsilon^i}^{\varepsilon^i}\right) = 0 \\ (\delta_\varepsilon - |\alpha| + 1)R_\alpha^\varepsilon - \frac{1}{2}\sum_{i=1}^n C_{\alpha^i}^\varepsilon + \frac{1}{2}\sum_{i=1}^n \xi_\varepsilon^i D_{\alpha_\varepsilon^i}^{\varepsilon^i} = 0. \end{cases}$$

Thus,

$$\dim H^1(\mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} \binom{n+k-r-1}{k-r},$$

where  $\binom{n}{2r+1}$  is the number of  $\varepsilon$  with  $|\varepsilon| = r + \frac{1}{2}$  and  $\binom{n+k-r-1}{k-r}$  is the number of  $\alpha$  with  $|\alpha| = k - r$ . □

For  $n = 2$ ,  $\dim H^1(\mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}) = 2\delta + 1$ .

## References

- [1] Agrebaoui B and Ben Fraj N, On the cohomology of the Lie superalgebra of contact vector fields on  $S^{1|1}$ , *Bell. Soc. Roy. Sci. Liège* **72**, 6, 2004, 365–375.
- [2] Arnal D, Ben Ammar M and Selmi M, Normalisation d’une représentation non linéaire d’une algèbre de Lie, *Annales de la faculté des sciences de Toulouse, 5<sup>e</sup> série, tome 9, n<sup>o</sup> 3, (1988), p 355–579*.
- [3] Basdouri I, Ben Ammar M, Cohomology of  $\mathfrak{osp}(1|2)$  acting on linear differential operators on the supercircle  $S^{1|1}$ . *Letters in Mathematical Physics* (2007) 81:239–251.
- [4] Ben Ammar M, Jabeur A, Safi I, cohomology of  $\mathfrak{osp}(1|2)$  acting on the space of bilinear differential operators on the superspace  $\mathbb{R}^{1|1}$ , *International Journal of Geometric Methods in Modern Physics* (2000).
- [5] Bouarroudj S, *Cohomology of the vector fields Lie algebras on  $\mathbb{RP}^1$  acting on bilinear differential operators*, *International Journal of Geometric Methods in Modern Physics* (2005), **2**; N 1, 23-40.
- [6] Fuchs D B, *Cohomology of infinite-dimensional Lie algebras*, Plenum Publ. New York, 1986.
- [7] Gargoubi H, *Sur la géométrie de l’espace des opérateurs différentiels linéaires sur  $\mathbb{R}$* , *Bull. Soc. Roy. Sci. Liège. Vol. 69, 1, 2000, 2147*.
- [8] Gargoubi H, Mellouli N and Ovsienko V *Differential operators on supercircle: conformally equivariant quantization and symbol calculus*, *Letters in Mathematical Physics* (2007) **79**:5165.
- [9] Lecomte B P A, *On the cohomology of  $\mathfrak{sl}(n+1; \mathbb{R})$  acting on differential operators and  $\mathfrak{sl}(n+1; \mathbb{R})$ -equivariant symbols*, *Indag. Math. NS. 11 (1), (2000), 95–114*.
- [10] A. Nijenhuis, R.W. Richardson, *Cohomology and deformations in graded Lie algebras*. *Bull. Amer. Math. Soc.*, 72 (1966), 1-29.